MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

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Problem Set 2

1. Let $B = [0,1]\backslash\mathbb{Q}$, the set of irrationals. Given $\epsilon > 0$, construct a closed set $F \subset B$ such that $\mu^* F \geq 1 - \epsilon$, equivalently $\mu^*(B\backslash F) \leq \epsilon$.

2. Prove the following: Let μ^* be the L-outer measure
   (a) μ^* is translation invariant.
   (b) $\mu^*(A + B) = \mu^*(B)$ if $\mu^*(A) = 0$.

3. Recall (state) Borel- Cantelli lemma.
   (a) Produce an example to show that the condition $\mu E_1 < \infty$ cannot be dropped.
   (b) Also, if $\Sigma \mu E_n = \infty$ with $\mu E_n < \infty$, $\forall n$, then it is not necessarily true that $\mu\left(\overline{\lim} E_n\right) = 0$.

4. (a) Let $\mu^* A > 0$, $A \subset \mathbb{R}$, then show that there is a non-measurable set contained in A (Hint: you may use the non-measurable set N).
   (b) Prove that any measurable subset E of the measurable N has outer measure 0. Is the result true for other non-measurable sets? Justify?

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $\{x : \ f(x) \geq r\}$ is measurable for all $r \in \mathbb{Q}$. Show that f is measurable.

6. Let $f : \mathbb{R} \to \mathbb{R}$, and $\{x : \ f(x) = \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$. Construct an example to show that f need not be measurable.

7. Let $f : [a,b] \to [-\infty, \infty]$ is measurable and f takes the values $\pm\infty$ only on a set of measure 0. Show that, for any $\epsilon > 0$, $\exists$ an $M > 0$ and a set $E \subset [a,b]$ such that $\mu E \leq \epsilon$ and $|f(x)| \leq M$, for $x \in E^c = [a,b]\backslash E$.

8. (a) Let E_n be a sequence of measurable sets, show that $\mu\left(\underline{\lim} E_n\right) \leq \underline{\lim}\, \mu E_n$.
   (b) Also prove $\mu\left(\overline{\lim} E_n\right) \leq \mu\left(\overline{\lim} E_n\right)$ if $\mu\left(\bigcup E_n\right) < +\infty$. The result need not be true if $\mu\left(\bigcup E_n\right) = +\infty$.

9. (a) (Generalized Cantor set): Construct a Cantor-type set $F \subset [0,1]$ by removing the middle intervals of length $\alpha/3^n$, $0 < \alpha < 1$ at the stage n. Show that F is closed, F^c is dense in $[0,1]$ and $\mu F = 1 - \alpha$ (in the standard cantor set, $\alpha = 1$).

   (b) Prove $C + C = [0,2]$.

10. Define Lipschitz continuity. Prove $f(x) = x^{1/2}$ is Lipschitz in $[\delta, \infty)$ for any $\delta > 0$, but f is not Lipschitz in $(0, \delta)$ for any $\delta > 0$.

11. Let $f : E \to \mathbb{R}$ be Lipschitz and let $A \subset E$ be such that $\mu A = 0$, then show that $\mu\left(f(A)\right) = 0$.

12. Construct two examples (with proof) to show that the conditions in Egoroff's theorem cannot be dropped; that is

   (a) $\mu E < \infty$ cannot be dropped.

   (b) In general, one cannot choose $\mu A = 0$.

   (c) Assuming Egroff's theorem is true for $f_n \to f$ everywhere on E, deduce that the theorem is true if $f_n \to f$ a.e. on E.

13. Let f_n be a sequence of measurable functions. Show that $\underline{\lim} f_n$ and $\overline{\lim} f_n$ are measurable. Also, if f and g are measurable, show that fg is measurable.

14. Let $f : E \to \mathbb{R}$ be measurable and $f = g$ a.e. Show that g is measurable.

15. Prove that $f^{-1}(U)$ is measurable (respectively, $\mathcal{B}$- measurable) for any Borel set U if f is measurable (respectively, $\mathcal{B}$-measurable) (Hint: may consider the class $\{U : \; f^{-1}(U) \text{ measurable}\}$)

16. Construct an example to show that $f \circ g$ need not be measurable if both f and g are measurable.